Topology and its Applications 13 (1982) 283–293
North-Holland Publishing Company

ON THE SMALL DIAGONALS

Zhou HAO-XUAN
Department of Mathematics, Sichuan University, Chengdu, China

Received 3 October 1980
Revised 5 February 1981

In this paper we prove that each Hausdorff compact (resp. T_3 countably compact) space with a small (resp. regularly small) diagonal is metrizable under CH+FA. Several 'nice' spaces, which have a small diagonal, but no G_δ-diagonal, are given under MA+ ¬CH. Some applications to the metrization problem are also obtained.

AMS Subj. Class.: Primary 54E35, 54E45; Secondary 54D20, 54D40, 54B05
Hušek problem small diagonal G_δ-diagonal Martin's Axiom MA
Zippin space Lindelöf space Fleissner's Axiom FA

1. Conventions and notations

1.1. Definition. A space is said to have a (regularly) small diagonal, if for any uncountable subset A of $X\times X\backslash\Delta$, where Δ is the diagonal of X, there is a (closed) neighborhood W of Δ, such that $|A\backslash W|>\omega$.

This is equivalent to saying that X does not have a ω_1-accessible diagonal [11]. Equivalently, one can use a G_δ-subset containing Δ, instead of W in Definition 1.1.

1.2. Definition. X is said to be an ω_1-like space, if there is a perfect onto mapping $f: X\to\omega_1$, such that each preimage is metrizable.

In the following, Δ_X, Δ_Y, Δ_Z always denote the diagonal of spaces X, Y, Z respectively, and k, λ, τ are cardinals.

1.3. Notation. For any X, define $\hat{\Delta}(X)=\min\{k$: there is a cofinal collection $\mathscr{U}$ of regularly open neighborhoods of Δ in $X\times X$, such that $|\mathscr{U}|=k\}$.

Here $\mathscr{U}$ cofinal, means that for any regularly open neighborhood V of Δ, there is $U\in\mathscr{U}$ such that $U\subseteq V$.

The definitions of the cardinal functions used in this paper can be found in [8]. We mention that d, L, t, s, w, χ, and ψ denote respectively density, Lindelöf degree, tightness, hereditary cellularity, weight, character, and pseudocharacter.

2. The main theorems

It is well-known that every T_2 compact (even countably compact [3]) space with a G_δ-diagonal is metrizable. This raises the following question: under what diagonal conditions are compact spaces metrizable? "To have a small diagonal" is a candidate, which is weaker than "to have a G_δ-diagonal." Another possible candidate is $t(\Delta, X^2) \leq \omega$, which is defined by $t(f(\Delta), X^2/\Delta) \leq \omega$ in the quotient space X^2/Δ (the only non-singleton equivalence class is Δ), where f is the projection. Note that a compact T_2 space X has a G_δ-diagonal, if and only if $\chi(f(\Delta), X^2/\Delta) \leq \omega$. When $t(X^2) \leq \omega$, one can prove that $t(\Delta, X^2) \leq \omega$ is weaker than X has a small diagonal by using a characterization theorem [1, (2.2.13)] in the space X^2/Δ. But, by using a Kunen line, a T_2 compact non-metrizable space X can be constructed such that $s(X^2) = \omega$, hence $s(X^2/\Delta) \leq \omega$. It follows that $t(\Delta, X^2) \leq \omega$, where s is the hereditary cellularity function [8]. It is natural to ask:

Hušek problem. Is every T_2 compact space with a small diagonal metrizable?

In [11], M. Hušek obtained a partial solution under CH (i.e. Fact 2.3). E. van Douwen proved any compact (D. Lutzer proved for Lindelöf) LOTS with a small diagonal is metrizable [11]. The aim of this paper is to give a consistent affirmative solution to the Hušek problem, and of another similar problem for countably compact spaces. The other direction seems harder, because only a few 'nice' spaces are known, which have small diagonals and no G_δ-diagonals. No such examples are known for Lindelöf spaces and countably compact spaces.

2.1. Notation [7]. Let FA be "If X is first countable, and contains the ordinal ω_1 as a subspace, then $\chi(\omega_1, X) = \omega_1$."

In [7], W. Fleissner proved Con(ZFC + $\exists$ an inaccessible ordinal) $\Rightarrow$ Con(ZFC + GCH + FA). Later F. Tall proved that one doesn't need the existence of inaccessible ordinals for a similar case [14]. Unfortunately, the reverse of the above implication is true, which follows from the proof of Solovay theorem [5, p. 213]. The argument is as follows: If ω_2 is accessible in L, then there is $A \subseteq \omega_1$ and $\omega_1 = \omega_1^{L[A]}$, $\omega_2 = \omega_2^{L[A]}$. But $\models_{L[A]} \diamondsuit^+$ [5], and $\diamondsuit^+ \Rightarrow \neg$FA [7], so the counterexample of Fleissner's is a real counterexample in ZFC. Recently F. Tall conjectured PFA $\Rightarrow$ FA, where PFA is the proper forcing axiom. It would follow that Con(MA + $\neg$CH + FA).

Before we prove the main theorems, let us mention and prove some facts and lemmas.

2.2. Fact. *The property of having a small diagonal is hereditary.*

2.3. Fact [7] (CH). *A T_2 compact space X is metrizable iff it has a small diagonal and one of the following holds:*

(1) $d(X) \leq \omega$,
(2) $t(X) \leq \omega$,
(3) $w(X) \leq \omega_1$ or $|X| \leq \omega_1$,
(4) $|C(x)| \leq \omega_1$.

We have a generalization of 2.3, whose proof needs the following lemma.

Lemma [8] (CH). *Let X be a Lindelöf space with $w(X \times X) \leq \omega_1$. Then $\chi(\Delta_X, X \times X) \leq \omega_1$. In particular, if X is a compact Hausdorff space with $|X| \leq \omega_1$, then $\chi(\Delta_X, X \times X) \leq \omega_1$.*

2.4. Lemma (CH). *Suppose X is a Lindelöf space with a small diagonal and one of the following holds.*
(1) $w(X) \leq \omega_1$,
(2) $d(X) \leq \omega$,
(3) $\chi(X) \leq \omega$,
then X has a G_δ-diagonal.

Proof. (1) Since X is Lindelöf and $w(X) \leq \omega_1$, it is easy to see by the above Lemma that $\chi(\Delta, X \times X) \leq w(X) \cdot L(X) \leq \omega_1$. Let $\mathcal{U} = \{U_\alpha : \alpha < w_1\}$ be a local base at Δ. If X does not have a G_δ-diagonal we can find $\{x_\alpha : \alpha < \omega_1\} \subseteq X \times X \backslash \Delta$, such that $x_\alpha \in \bigcap_{\xi \leq \alpha} U_\alpha \backslash \{x_\beta : \beta < \alpha\}$. It contradicts the fact that X has a small diagonal.

2.5. Remark. We do not know if the condition (3) can be replaced by $\psi(X) \leq \omega$. We only can show that if a T_2 space X has a small diagonal and $L(X) \leq \omega$, $|X| \leq \omega_1$, then $\psi(X) \leq \omega$.

2.6. Lemma. *Any ω_1-like space is first countable, countably compact, and hence sequentially compact.*

2.7. Lemma (CH). *If a T_2 space X satisfies*:
(*) *For any countably compact subspace Y' of size $\leq \omega_1$, $\chi(\Delta_{Y'}, Y' \times Y') \neq \omega_1$.*
(**) *Any separable subspace is metrizable.*
Then if Y is a countably compact non-metrizable subspace of size $\leq \omega_1$, Y contains an ω_1-like subspace.

Proof. By (**), we have $d(Y) > \omega$, and Fact 2.3 implies that Y cannot be compact, hence not Lindelöf. Thus there is a subset $D = \{x_\alpha \in Y : \alpha < \omega_1\}$ such that for all $\alpha < \omega_1$, $x_\alpha \notin \mathrm{Cl}_Y\{x_\beta : \beta < \alpha\}$, and $\mathrm{Cl}_Y D$ is non-compact too. Let $X_\delta = \mathrm{Cl}_Y\{x_\alpha : \alpha < \delta\}$ and $Z = \bigcup_{\delta < \omega_1} X_\delta$. Since every X_δ is separable, it is compact and metrizable by (*); hence Z is countably compact and non-compact by (*) and the lemma following 2.3. Assume $G = \{G_\alpha : \alpha < \omega_1\}$ is an open cover of Z without any countable subcover.

Inductively define $z_\alpha \in Z(\alpha<\omega_1)$, $f, g \in {}^{\omega_1}\omega_1$ such that for each $\alpha<\omega_1$, we have

(i) $\{z_\beta : \beta<\alpha\} \subseteq X_{f(\alpha)}$,

(ii) $X_{f(\alpha)} \subseteq \bigcup\{G_\beta : \beta<g(\alpha)\}$,

(iii) $z_\alpha \in Z \setminus \bigcap\{G_\beta : \beta<g(\alpha)\}$.

It is not difficult to check that $\{z_\alpha : \alpha<\omega_1\}$ is a free ω_1-sequence in Z, which means $\forall\alpha<\omega_1$,

$$\mathrm{Cl}_Z\{z_\delta : \delta\leq\alpha\} \cap \mathrm{Cl}_Z\{z_\delta : \delta>\alpha\} = \emptyset.$$

Let $S_\alpha = \mathrm{Cl}_z\{z_\beta : 0\leq\beta\leq\alpha\}$. Then each S_α is a compact metrizable space by (**), and is both open and closed in S, where $S=\bigcup_{0\leq\alpha<\omega_1} S_\alpha$. Also note that S is countably compact.

Define $f: S\to\omega_1$ by $f(x)=\min\{\alpha : X\in S_\alpha\}$. To see that f is continuous, let $x\in S$, let $\alpha=f(x)$, and let $(\beta,\alpha]$ be an open neighborhood of α. Then $(S_\alpha - S_\beta)$ is an open neighborhood of x and $f(S_\alpha - S_\beta)\subseteq(\beta,\alpha]$. The function f is closed since it is continuous, S is countably compact, and ω_1 is first countable and Hausdorff. Finally, for all $\alpha<\omega_1$, $f^{-1}(\{\alpha\}) = (S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta)$, a compact metrizable space. Thus S is an ω_1-like space.

2.8. Lemma (FA). *If S is an ω_1-like space, then $\chi(\Delta_S, S\times S)=\omega_1$.*

Proof. Let f witnesses the definition of ω_1-like spaces, and $\alpha<\omega_1$. Let $T_\alpha = f^{-1}(\{\alpha\})$, $S_\alpha = f^{-1}(\alpha+1)$, $\Delta_\alpha = \Delta_{T_\alpha}$. In $S\times S$, since Δ_α is compact, and contained in a compact metrizable clopen subspace, for example $S_\alpha\times S_\alpha$, holds $\chi(\Delta_\alpha, S\times S)=\omega$.

Define $h: S\times S\to Q=(S\times S\setminus\Delta_S)\cup\omega_1$ as follows: For $s\in S\times S\setminus\Delta_S$, let $h(s)=s$, for $x\in\Delta_\alpha$, let $h(x)=\alpha$. Assume Q possesses the quotient topology induced by h.

Claim 1. $h(\Delta_S)$ is homeomorphic with the space ω_1 of countable ordinals. (Here I hope the reader knows in which cases ω_1 represents the ordered space.)

Define $\varphi: h(\Delta_S)\to\omega_1$ such that $\forall\alpha<\omega_1$, $\varphi(h(\Delta_\alpha))=\alpha$. Assume $\beta\in\lim\omega_1$, and G is an open neighborhood of Δ_β in Δ_S. Let E be an open subset in $S\times S$ such that $E\cap\Delta_S=G$. Since $\Delta_\beta\subseteq E$, there is a neighborhood $U(x)$ for each $x\in\Delta_\beta$ such that

$$U=\cup\{U(x)\times U(x) : x\in T_\beta\}\subseteq E.$$

$\{U(x): x\in T_\beta\}$ is an open cover of T_β in S, hence there is $\delta<\beta$, such that for all $\alpha, \delta<\alpha<\beta$, $T_\alpha\subseteq\bigcup\{U(x): x\in T_\beta\}$. Thus $\Delta_\alpha\subseteq\bigcup\{U(x)\times U(x);\ x\in T_\beta\}$ and $\Delta_\alpha\subseteq G$. It means that φ^{-1} is continuous. Conversely, φ is continuous because $\forall\alpha<\omega_1$, $h^{-1}\circ\varphi^{-1}(\alpha)=\bigcup_{\beta<\alpha}\Delta_\beta$ is open in Δ_S, which follows $\varphi^{-1}(\alpha)$ is open in $h(\Delta_S)$.

Claim 2. h is closed.

It is easy to check that $S\times S$ is countably compact and the quotient space Q is first countable, hence h is a closed mapping, in fact, if $\{E_n : n<\omega\}$ is a countable local base of Δ_β in $S_\beta\times S_\beta$, then $\{h(E_n): n<\omega\}$ is a local base of $h(\Delta_\beta)$ in Q, because $(E_n\setminus\Delta_s)\cup\bigcup_{\delta<\alpha\leq\beta}\Delta_\alpha\subseteq h(E_n)$ is an open saturated subset of $S_\beta\times S_\beta$ by the proof of Claim 1, where $\cup_{\delta<\alpha<\beta}\Delta_\alpha\subseteq E_n$ and a set W is *saturated* provided $W=h^{-1}h(W)$.

Finally, since FA implies $\chi(h(\Delta_S), Q)=\omega_1$, and h is a closed mapping, we have $\chi(\Delta_S, S\times S)\leqslant\omega_1$. By Chaber's Theorem [3], $\chi(\Delta_S, S\times S)\neq\omega$, so $\chi(\Delta_S, S\times S)=\omega_1$.

2.9. Theorem (CH+FA). *For arbitrary compact T_2 space X, the following are equivalent.*

(1) *X is metrizable.*

(2) *X has a small diagonal.*

(3) *For any countably compact subspace Y of X, $\chi(\Delta_Y, Y\times Y)\neq\omega_1$.* [1]

(4) *X satisfies (*) and (**).*

(5) *Any countably compact subspace Y of size $\leqslant\omega_1$ is metrizable.*

(6) *Any subspace of size $\leqslant\omega_1$ is metrizable.*

Proof. (1)⇒(2). Trivial.

(2)⇒(3). Use Chaber's theorem, similar to the proof of 2.4.

(3)⇒(4). For any separable subset Z, since $d(\bar{Z})\leqslant\omega$, we have $w(\bar{Z})\leqslant\omega_1$. hence $\chi(\Delta_{\bar{Z}}, \bar{Z}\times\bar{Z})\leqslant\omega_1$ (see [8]). Therefore (3) implies $\chi(\Delta_{\bar{Z}}, \bar{Z}\times\bar{Z})\leqslant\omega$, and Z is metrizable. Here $\bar{Z}=\mathrm{Cl}\,Z$.

(4)⇒(5). Follows from Lemma 2.7, 2.8.

(5)⇒(6). For any subset $Y\subseteq X$, $|Y|\leqslant\omega_1$, one can inductively construct a countably compact subspace $Z\supseteq Y$, and such that $|Z|\leqslant\omega_1$. Thus it follows from (5).

(6)⇒(1). If $hL(X)>\omega$ (i.e. X is not hereditarily Lindelöf), X will contain a right-separated subset D of size ω_1. Assume D' is a countably compact subspace of size ω_1, and $D\subseteq D'$; but this is impossible by (6), hence $|X|\leqslant 2^{hL(X)}\leqslant\omega_1$, and X is metrizable.

2.10. Theorem (CH+FA). *For arbitrary countably compact T_3 space X, the following are equivalent.*

(1) *X is metrizable.*

(2) *X has a regularly small diagonal.*

(3) *For any countably compact subspace Y, $\hat{\Delta}(Y)\neq\omega_1$.*

(4) *X satisfies (**) and (***): for any countably compact subspace Y of size ω_1, $\hat{\Delta}(Y)\neq\omega_1$.*

(5) *Any countably compact subspace Y of size $\leqslant\omega_1$ is metrizable.*

(6) *Any subspace of size $\leqslant\omega_1$ is metrizable.*

Proof. (1)⇒(2). Trivial.

(2)⇒(3). Almost the same as Theorem 2.9.

(3)⇒(4). Only prove (**). Suppose Z is a separable subset, we can assume Z to be closed, hence countably compact. Since $Z\times Z$ is separable, $|\mathrm{RO}(Z\times Z)|\leqslant\omega_1$, where $\mathrm{RO}(Z\times Z)$ is the collection of all regularly open subsets in $Z\times Z$, hence

[1] Actually, we can use (4)′: X satisfies (*), instead of (4) in the next clause. The same holds for Theorem 2.10

$\hat{\Delta}(Z) \leq \omega_1$, and $\hat{\Delta}(Z) \leq \omega$ because of (3). On the other hand, since Δ_Z is the intersection of its regularly open neighborhoods, by Chaber's Theorem, Z is metrizable.

(4)$\Rightarrow$(5) (Sketch). We need strengthened Lemma 2.7′ and 2.8′. Instead of $\chi(\Delta_{Y'}, Y' \times Y') \neq \omega_1$ (in Lemma 2.7) and $\chi(\Delta_S, S \times S) = \omega_1$ (in Lemma 2.8), we use $\hat{\Delta}(Y') \neq \omega_1$ and $\hat{\Delta}(S) = \omega_1$ respectively. Only Lemma 2.8′ need to be considered, but it is easy to check because $\hat{\Delta}(S) \leq \chi(\Delta_S, S \times S) \leq \omega_1$, and from $\hat{\Delta}(S) = \omega$ it follows that S is metrizable, hence not ω_1-like.

(5)$\Rightarrow$(6). The same as Theorem 2.9.

(6)$\Rightarrow$(1). The same as Theorem 2.9.

These theorems are so heavily dependent on the additional hypotheses, so they usually are not true. This is illustrated by the subsequent examples.

2.11. Remark. CH is equivalent to "For any compact T_2 space X, X is metrizable iff X has the property (5) in Theorem 2.9." To see this, we assume $\neg$CH and construct a space which satisfies (5), but is not metrizable. Consider the Alexandroff's double circles space X. Since X is first countable, every countably compact subspace is compact. Besides, X omits all cardinals between ω and 2^ω (i.e. X doesn't have compact subspaces of those sizes).

2.12. Remark. It can be observed that the implication (2)$\Rightarrow$(3) can not be reversed under $\neg$CH.

Let X be the Two Arrow space. It is a compact LOTS, and a first countable space. By the same reason, all countably compact subspaces are compact. Let $Y \subseteq X$ be a closed subset. It can only happen that $|Y| = \omega$ or $|Y| = 2^\omega$. If $|Y| = \omega$, obviously $\chi(\Delta_Y, Y \times Y) \neq \omega_1$. If $|Y| = 2^\omega$, then $w(Y) = 2^\omega$, and $\chi(\Delta_Y, Y \times Y) \geq 2^\omega$. But, by E. van Douwen's result mentioned above, X does not have a small diagonal.

2.13. Problem. Are (1) and (2) of Theorem 2.9 (resp. Theorem 2.10) equivalent without CH?

3. Further results and applications

3.1. Lemma (CH). *Let X be a T_2 compact space having the properties* (*), (**) *(hence if having a small diagonal) and $|X| \leq \omega_2$, then X is metrizable.*

Proof. Suppose X is not metrizable. By (**), X is not separable. As in the proof of 2.7, one can construct an ω_1-free sequence $\{x_\alpha : \alpha < \omega_1\}$ in X. Clearly, if we denote $X_\alpha = \mathrm{Cl}_X\{x_\delta : \delta < \alpha\}$, $Y = \bigcup_{\alpha < \omega_1} X_\alpha$, then X_α's are clopen subsets of Y and Y is an ω_1-like subspace. Without loss of generality, assume $X = \mathrm{Cl}\, Y$. Let $Z =$

$\bigcap_{\alpha<\omega_1}(X\setminus X_\alpha)$, where each $X\setminus X_\alpha$ is clopen in X. Since $\chi(Z, X)=\omega_1$,

$$\chi(z, X)\leqslant\chi(z, Z)\cdot\chi(Z, X)\leqslant\omega_1\cdot\chi(z, Z)$$

holds for each $z\in Z$. Moreover we have $\chi(z, X)\geqslant\omega_1$ for all $z\in Z$ because $\operatorname{Int} Z=\emptyset$.

But, since X has no convergent ω_1-sequences, (for, if $\{x_\alpha:\alpha<\omega_1\}$ convergents to x, let

$$S=\operatorname{Cl}_X(\{x_\alpha:\alpha<\omega_1\}\cup\{x\})=\bigcup_{\alpha<\omega_1}\operatorname{Cl}_X\{x_\beta;\beta<\alpha\}\cup\{x\},$$

we will have $\chi(\Delta_S, S\times S)\leqslant\omega_1$ and $|S|\leqslant\omega_1$. The reason has been shown just at the beginning of our proof, hence S is metrizable by (*), which is a contradiction), $\chi(z, X)=\omega_1$ can not happen. Thus for every $z\in Z$, $\chi(z, Z)\geqslant\omega_2$. By the Pošpisil–Čech theorem, we have $|Z|\geqslant\omega_3$. This is a contradiction.

3.2. Corollary (CH). *If the T_2 compact, non-metrizable space X has a small diagonal, then X omits ω_2. (For the definition, see* [8].)

Now we mention a couple of facts, that have independent interest.

3.3. Fact (CH). *If X is a T_2 compact space, then X is metrizable iff each continuous image of X has a small diagonal.*

Proof. Use the same notation as Lemma 3.1. Assume that X is not metrizable. Obviously X has a small diagonal. By the proof of Lemma 3.1, there exists an ω_1-like subspace Y and a perfect mapping $f:\operatorname{Cl}_X Y\to\omega_1+1$, f can be continuously extended to $f':X\to Z$, where $Z=(X\setminus\operatorname{Cl}_X Y)\cup(\omega_1+1)$ is the evident quotient space, induced by f'; and define $f'(\operatorname{Cl}_X Y\setminus Y)=\{\omega_1+1\}$. It is not hard to show that the subspace ω_1+1 of Z is homeomorphic with the ordered space ω_1+1. But ω_1+1 doesn't have a small diagonal, so does not Z.

3.4. Fact (GCH). *A T_2 compact space X is metrizable iff X has the following property.*

(I) *For any subset $A\subseteq X\times X\setminus\Delta$, where $|A|=\omega_1$* or ω_2, *there is a neighborhood W of Δ such that $|A\setminus W|=|A|$.*

Proof. Since X has a small diagonal, if X is not metrizable, there will exist a convergent ω_2-sequence in X by the proof of Lemma 3.1. In fact, following the notations in Lemma 3.1, each $y\in\operatorname{Cl}_X Y\setminus Y$ has character $\geqslant\omega_1$ because $\operatorname{Int}(\operatorname{Cl}_X Y\setminus Y)=\emptyset$. If $\forall y\in Y$, $\chi(y, \operatorname{Cl}_X Y)\leqslant\omega_1$, then $|\operatorname{Cl}_X Y|\leqslant\omega_2$ by Arhangel'skij's Theorem, hence X is metrizable by Lemma 3.1. It is not possible. Let $y\in\operatorname{Cl}_X Y$ such that $\chi(y, \operatorname{Cl}_X Y)=\omega_2$. It is easy to find out an ω_2-sequence $\{y_\alpha:\alpha<\omega_2\}$, which is convergent to y. Let $A=\{\langle y_\alpha, y\rangle:\alpha<\omega_2\}$, there is no neighborhood W of Δ such that $|A\setminus W|=|A|$. This contradicts the assumption.

3.5. Fact (CH). *A T_2 compact space X is metrizable iff X has the following property.*

(II) *For any uncountable collection $\{(F_\alpha, x_\alpha)\}$, where F_α's are compact, $x_\alpha \notin F_\alpha$, then there is a neighborhood W of Δ, such that there are uncountably many α such that $\{(y, x_\alpha): y \in F_\alpha\} \cap W = \emptyset$.*

Proof. If X is metrizable, let $\varepsilon_\alpha = \rho(x_\alpha, F_\alpha)$, where ρ is the metric. There is $\varepsilon > 0$ such that $|\{\alpha: \varepsilon_\alpha > \varepsilon\}| > \omega$. Let W be the ε-neighborhood of Δ.

Conversely, since X has a small diagonal, we can assume $t(X) > \omega$. Let $\{x_\alpha : \alpha < \omega_1\}$ be a free ω_1-sequence, and $Y = \bigcup_{\alpha<\omega_1} X_\alpha$ be an ω_1-like space, where $X_\alpha = \mathrm{Cl}\{x_\beta : \beta < \alpha\}$.

Let $f: Y \to \omega_1$ be the perfect mapping, and $F_\alpha = f^{-1}[\alpha+1, \omega_1)$. It follows $x_\alpha \in f^{-1}(\alpha)$, $x_\alpha \notin F_\alpha$. By the pressing down lemma it is not difficult to show there is no neighborhood W of Δ such that

$$|\{\alpha : F_\alpha \times \{x_\alpha\} \cap W = \emptyset\}| > \omega.$$

It can be shown that a space has the property (II) iff the hyperspace $\mathscr{C}(X)$ has a small diagonal.

In [11], M. Hušek asked if every ω_1-compact space with a small diagonal has a G_δ-diagonal. The answer turns out to be negative by the following examples under MA + $\neg$CH, hence the Theorem 1 of [11] is a real generalization. First we observe a fact.

3.6. Fact (MA + $\neg$CH). *Let $\mathbb{P}$ be the irrationals, $\mathbb{Q}$ the rationals. For any uncountable subset $\mathbb{P}' \subseteq \mathbb{P}$, there is an open subset U such that $\mathbb{Q} \subseteq U$, and $|\mathbb{P}' \setminus U| > \omega$.*

Proof. Arbitrarily take a subset $\mathbb{P}_1 \subseteq \mathbb{P}'$ such that $|\mathbb{P}_1| = \omega_1$. By the theorem in [12, § 4.2], $\mathbb{P}_1$ is a meager set, and after slightly changing the proof of Martin–Solovay's Theorem, we can get a G_δ-set $\bigcap_{n<\omega} U_n \supseteq \mathbb{Q}$, and $\mathbb{P}_1 \cap \bigcap_{n<\omega} U_n = \emptyset$. It follows that there is a $n < \omega$ such that $|\mathbb{P}_1 \setminus U_n| \geqslant \omega_1$.

3.7. Example. It is easy to verify that the space Y of Example 1 in [13] has a small diagonal by use of Fact 3.6, hence Y is a hereditarily paracompact space with a small diagonal, with no G_δ-diagonal under MA + $\neg$CH.

3.8. Example. In the same way, by virtue of Fact 3.6 we can show the space Y of Example 3 in [13] has a small diagonal, hence under MA + $\neg$CH there is a T_2 hereditarily Lindelöf space Y with a small diagonal, with no G_δ-diagonal.

M. Hušek [11] asked whether there is an ω_1-compact space X such that $\omega_1 \in \Delta X \setminus \bar{\Delta} X$? (For the terms, see [11].) The following example answers this question consistently.

3.9. Example (MA + $\neg$CH). Let $\mathbb{R}$ be the real line with the usual topology. Consider a new point $\infty \notin \mathbb{R}$, let the class of all open dense subsets be the local base at ∞. Then $Z = \mathbb{R} \cup \{\infty\}$ is an ω_1-compact space with a small diagonal, but not a regularly small diagonal, hence $\omega_1 \in \Delta Z \setminus \tilde{\Delta} Z$.

3.10. Example (MA + $\neg$CH). Let $\mathscr{C}$ consist of all Cauchy sequences in $\mathbb{P}$, each of which converges to certain non-dyadic rational r, i.e. r does not have the form $k \cdot 2^{-n}$, $k, n \in N$. Denote the subset consisting of elements of the Cauchy sequence s by s^*. Define an equivalent relation $\sim$ in $\mathscr{C}$ as: $s \sim t$ iff $s^* = t^*$, and $\tilde{s}$ is the equivalent class determined by s. By Zorn's Lemma, there is a maximal subfamily $\tilde{\mathscr{C}}$ of $\{\tilde{s}; s \in \mathscr{C}\}$, such that:

(i) if $\tilde{s} \neq \tilde{t}$, then $|s^* \cap t^*| < \omega$, and

(ii) for each non dyadic rational r, there is $\tilde{s} \in \tilde{\mathscr{C}}$ such that s converges to r.

Let $X = \mathbb{P} \cup \tilde{\mathscr{C}}$, and the topology is defined as follows: let each irrational be open; for $\tilde{s} \in \tilde{\mathscr{C}}$, if s converges to a rational $r \in (k/2^n, (k+1)/2^n)$, let $U_n(\tilde{s}) = \{s^* \cap P_{nk}\} \cup \{\tilde{s}\}$, where

$$P_{nk} = \mathbb{P} \cap (k/2^n, (k+1)/2^n), \quad n, k, \in N.$$

$\{U_n(\tilde{s}); n < \omega\}$ is the local base at $\tilde{s}$. X is non-developable.

By the same method in [2], let X_1, X_2 be two copies of X, and identify each point in $\tilde{\mathscr{C}}_i$ $(i = 1, 2)$. The corresponding quotient space Y is a T_2 locally compact space with a small diagonal, with no G_δ-diagonal by virtue of Fact 3.6. We have a locally compact, normal, non-metrizable space with a small diagonal under $E(\omega_2) + \diamondsuit_2(E)$ [10, Theorem 2.9]. I do not know if it has a G_δ-diagonal.

Another easy, but less interesting example is $(2^{\omega_2})\omega_2$, which has a small diagonal, and without G_δ-diagonal (see [4, § 15]). The following question is open.

3.11. Problem. Does every T_3 Lindelöf space with a small diagonal have a G_δ-diagonal?

Finally, we give two applications to the metrization theory.

3.12. Theorem (CH+FA). *If X is a Lindelöf Zippin space (i.e. it has a T_2 compactification with a countable remainder), then X is metrizable iff it has a small diagonal.*

Proof. Suppose bX is a compactification of X such that $|bX \setminus X| \leq \omega$. We are going to prove bX has a small diagonal if X has.

Let $bX \setminus X = \{z_i: i < \omega\}$, A an uncountable subset $bX \times bX \setminus \Delta_{bX}$, $|A| = \omega_1$. Since X is Lindelöf, each subset of size ω_1 has a completely accumulation point. Consider

$$A_{1i} = \{(x, z_i) \in A: x \in bX\}, \qquad A_{2i} = \{(z_i, x) \in A: x \in bX\},$$

where $i<\omega$. If $|A_{1i}|\geqslant\omega_1$, let (y,z_i) be an ω_1-accumulation point of $A_{1i}\backslash(b^*X\times b^*X)$ in $X\times\{z_i\}$. Since bX is T_2, we have a neighborhood W of Δ_{bX} such that $(y,z_i)\notin \mathrm{Cl}_{bX\times bX}W$, hence $|A\backslash W|\geqslant\omega_1$, the same argument is valid for A_{2i}.

Assume $\forall i<\omega$, $|A_{1i}\cup A_{2i}|\leqslant\omega$. Since $|A\cap X\times X|\geqslant\omega_1$, there is an uncountable subset $B\subseteq A\cap X\times X$ such that B is the graph of a certain function defined on a subset of X. In fact, if $|\pi_1(A\cap X\times X)|\leqslant\omega$, (here π_1 is the projection), then there is $x\in X$ such that

$$|\pi_1^{-1}(x)\cap[A\cap(X\times X)]|\geqslant\omega_1.$$

Since (x,x) cannot be the only ω_1-accumulation point of $\pi^{-1}(x)\cap A$ by the definition of small diagonal, there should be $y\neq x$ such that (x,y) is an ω_1-accumulation point of A. Therefore we can assume $|\pi_1(A\cap X\times X)|\geqslant\omega_1$. In each vertical line, pick a point belonging to A (if there is one), all of these consist of the desired subset B.

Since X has a small diagonal, there is a neighborhood U of Δ_x in $X\times X$ such that $|B\backslash U|\geqslant\omega_1$. Define $C=B\backslash U$, and take an ω_1-accumulation point $x\in X$ of π_1C, and two neighborhoods G_1, G_2 of x in bX, such that

$$x\in G_1\subseteq \mathrm{Cl}_{bX}G_1\subseteq G_2,\qquad G_2\times G_2\cap X\times X\subseteq U.$$

For each $z\in bX\backslash\{x\}$, let $G(z)$ be a neighborhood of z such that $G(z)\subseteq G_2$ if $z\in G_2$; and $G(z)\cap G_1=\emptyset$ if $z\notin G_2$.

Define $D=\{(x',y')\in C: x'\in G_1\cap X\}$. It is clear that the neighborhood

$$W=\cup\{G(z)\times G(z): z\in bX\backslash\{x\}\}\cup G_1\times G_1$$

is disjoint with D, and $|D|\geqslant\omega_1$.

Actually what we have proved is that if X is Lindelöf and $X\subseteq Y$, $|Y\backslash X|\leqslant\omega$, then X has a small diagonal iff Y has. Besides, it can be shown that if X is an open subset, or a G_δ-subset of Y, (with the same restriction for X, Y as above), then X has a G_δ-diagonal iff Y has. Maybe, this suggests to us a way to construct a Lindelöf space with a small diagonal and without G_δ-diagonal. Unfortunately, we don't know if the property to have a G_δ-diagonal is preserved by adding countably many new points to a Lindelöf space.

3.13. Corollary (CH). *Lindelöf, semi-stratifiable, non-metrizable spaces are not Zippin spaces.*

Similarly we have

3.14. Corollary (CH + FA). *Every locally compact, $[\omega_1,\omega_1]$ compact space (i.e. each subset of size ω_1 has an ω_1-accumulation point) with a small diagonal is metrizable.*

It is interesting to observe that E. van Douwen and H. Wicke [6] constructed (absolutely) a locally compact, locally countable, submetrizable (hence having a G_δ-diagonal), ω_1-compact space which is not normal, even without many weak covering properties.

Acknowledgements

The author is indebted to E. van Douwen for his help, and he also is thankful to the referee for his suggestions and corrections.

References

[1] A. Arhangel'skii, Construction and classification of topological spaces and cardinal invariants, YMH 33 (1978) 29–84.

[2] D.K. Burke, A nondevelopable locally compact Hausdorff space with a G_δ-diagonal. General Topology Appl. 2 (1972) 287–291.

[3] J. Chaber, Conditions which imply compactness in countably compact spaces, Bull. Acad. Pol. Sci. Ser. Math. 24 (1976) 993–998.

[4] W.W. Comfort and S. Negrepontis, The Theory of Ultrafilters (Springer-Verlag, Berlin and New York, 1974).

[5] K.J. Devlin, Aspects of Constructability (Springer-Verlag, Berlin and New York, 1973).

[6] E. van Douwen and H. Wicke, A real, weird topology on the reals, Houston J. Math. 3 (1977) 141–152.

[7] W. Fleissner, The character of w_1 in the first countable spaces, Proc. Amer. Math. Soc. 62 (1977) 149–155.

[8] I. Juhász, Cardinal functions in topology, Math. Centre Tracts 34, Amsterdam, 1971.

[9] I. Juhász, Two set-theoretic problems in topology, Proc. 4th Prague Top. Symp. 1976. Lecture Notes in Math., Vol. 609 (Springer-Verlag, Berlin, 1977) 115–123.

[10] I. Juhász, Consistency results in topology, in: J. Barwise, ed., Handbook of Mathematical Logic (North-Holland, Amsterdam, 1977) 503–522.

[11] M. Hušek, Topological spaces without $\aleph$-accessible diagonal, Comm. Math. Univ. Carolinae, 18 (1977) 777–788.

[12] D.A. Martin and R.M. Solovay, Internal Cohen extensions, Ann. of Math. Logic 2 (1970) 143–178.

[13] V. Popov, A perfect map need not preserve a G_δ-diagonal, General Topology Appl. 7 (1977) 31–33.

[14] F. Tall, Some applications of small cardinal collapse in topology, to appear in Proc. Budapest Top. Conf. 1978.